Last time: Basic notations + terminology.

Linear Systems of Equations

Defn: Let $x_1, x_2, \cdots, x_n$ be variable symbols (or variables).

A linear combination of these variables is any sum of form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

where $a_1, a_2, \cdots, a_n$ are constants (i.e. coefficients).

NB: Constants are real numbers.

A linear equation is an equation $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$ where a_i's and b are all constants. A linear system of equations (or linear system) is a collection of linear equations.

$$\begin{cases} a_{1,1} x_1 + a_{1,2} x_2 + \cdots + a_{1,n} x_n = b_1 \\ a_{2,1} x_1 + a_{2,2} x_2 + \cdots + a_{2,n} x_n = b_2 \\ \qquad \vdots \\ a_{m,1} x_1 + a_{m,2} x_2 + \cdots + a_{m,n} x_n = b_m \end{cases}$$

NB: This is an $m \times n$ system, or a system with m equations in n unknowns.

Ex: The system $\begin{cases} x - y + 2z = 0 \\ 3x + 0y + 4z = 4 \\ y + 0x - 2z = 2 \end{cases}$ is linear

$$x \leftrightarrow x_1, \quad y \leftrightarrow x_2, \quad z \leftrightarrow x_3$$

Non Ex: The system $\begin{cases} x^2 + y^2 = 4 \\ -y + x = 3 \end{cases}$ is not linear

Defn: A solution to an $m \times n$ linear system is an n-tuple (or vector) of constants satisfying all equations simultaneously.

Goal: Given a linear system, compute all solutions

Ex: Solve $\begin{cases} x - y + 2z = 0 \\ 3x + 4z = 4 \\ y - 2z = 2 \end{cases}$.

Sol: If (x, y, z) is a solution to this system:

$$\begin{cases} x - y + 2z = 0 \\ 3x + 4z = 4 \\ y - 2z = 2 \end{cases} \xrightarrow{Eq3 + Eq1 \to E1} \begin{cases} x = 2 \\ 3x + 4z = 4 \\ y - 2z = 2 \end{cases}$$

$$\xrightarrow{E2 - 3E1 \to E2} \begin{cases} x = 2 \\ 4z = -2 \\ y - 2z = 2 \end{cases} \xrightarrow{\frac{1}{4} E2 \to E2} \begin{cases} x = 2 \\ z = -\frac{1}{2} \\ y - 2z = 2 \end{cases}$$

$$\xrightarrow{E3 + 2E2 \to E3} \begin{cases} x = 2 \\ z = -\frac{1}{2} \\ y = 1 \end{cases} \rightsquigarrow \begin{cases} x = 2 \\ y = 1 \\ z = -\frac{1}{2} \end{cases}$$

$\therefore$ The system has solution $\begin{bmatrix} 2 \\ 1 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

NB: The above method is Gaussian elimination.

This method always solves a linear system.

IDEA: Systematic elimination of variables...

NB: Every linear system can be solved using the following three operations:

1. Swap two rows
2. Multiply a row by a nonzero constant.
3. Add two rows and replace one of them with the result.

These are the elementary (row) operations.

Ex: Solve the system
$$\begin{cases} 2x - 2y + z = 0 \\ 4y + z = 20 \\ x + z = 5 \\ x + y - z = 10 \end{cases}$$

NB: this 4×3 system is "overdetermined" because it has more equations than variables.

$$\begin{cases} 2x - 2y + z = 0 \\ 4y + z = 20 \\ x + z = 5 \\ x + y - z = 10 \end{cases} \xrightarrow{E1 \leftrightarrow E3} \begin{cases} x + z = 5 \\ 4y + z = 20 \\ 2x - 2y + z = 0 \\ x + y - z = 10 \end{cases}$$

$$\xrightarrow[E4 - E1 \to E4]{E3 - 2E1 \to E3} \begin{cases} x + z = 5 \\ 4y + z = 20 \\ -2y - z = -10 \\ y - 2z = 5 \end{cases} \xrightarrow[-E3 \to E3]{E4 \leftrightarrow E2} \begin{cases} x + z = 5 \\ y - 2z = 5 \\ +2y + z = +10 \\ 4y + z = 20 \end{cases}$$

$$\xrightarrow[E4 - 4E2 \to E4]{E3 - 2E2 \to E3} \begin{cases} x + z = 5 \\ y - 2z = 5 \\ 5z = 0 \\ 9z = 0 \end{cases} \xrightarrow[\frac{1}{9}E4 \to E4]{\frac{1}{5}E3 \to E3} \begin{cases} x + z = 5 \\ y - 2z = 5 \\ z = 0 \\ z = 0 \end{cases}$$

$$\xrightarrow[\substack{E2 + 2E3 \to E2 \\ E1 - E3 \to E1}]{E4 - E3 \to E4} \begin{cases} x = 5 \\ y = 5 \\ z = 0 \\ 0 = 0 \end{cases} \longrightarrow \begin{cases} x = 5 \\ y = 5 \\ z = 0 \end{cases}$$

Hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$ is the unique solution to this system ☒

Ex: Solve
$$\begin{cases} x - y + z = 2 \\ x + y - z = -1 \\ 3x + y - z = 1 \end{cases}$$

Sol: Applying Gaussian elimination:

$$\begin{cases} x - y + z = 2 \\ x + y - z = -1 \\ 3x + y - z = 1 \end{cases} \xrightarrow[E3 - 3E1 \to E3]{E2 - E1 \to E2} \begin{cases} x - y + z = 2 \\ 2y - 2z = -3 \\ 4y - 4z = -5 \end{cases}$$

$$\xrightarrow{E3-2E2\to E3}\begin{cases} x - y + z = 2 \\ 2y - 2z = -3 \\ 0 = 1 \end{cases}$$

$\leftarrow$ Contradiction.

Never true!

Hence this system has no solutions!

I.e its solution set is empty.

$$\begin{cases} x - y + z = 2 \\ 2y - 2z = -3 \\ 4y - 4z = -5 \end{cases}$$

$\boxed{}$

Ex: Solve the system

$$\begin{cases} 2x + z + w = 5 \\ y - w = -1 \\ 3x - z - w = 0 \\ 4x + y + 2z + w = 9 \end{cases}$$

Sol:

$$\begin{cases} 2x + z + w = 5 \\ y - w = -1 \\ 3x - z - w = 0 \\ 4x + y + 2z + w = 9 \end{cases} \xrightarrow[E3-E1\to E3]{E4-2E1\to E4} \begin{cases} 2x + z + w = 5 \\ y - w = -1 \\ x - 2z - 2w = -5 \\ y - w = -1 \end{cases}$$

$$\xrightarrow[E4-E2\to E4]{E1-2E3\to E1} \begin{cases} 5z + 5w = 15 \\ y - w = -1 \\ x - 2z - 2w = -5 \\ 0 = 0 \end{cases}$$

$$\xrightarrow[2E1+E3\to E3]{\frac{1}{5}E1\to E1} \begin{cases} z + w = 3 \\ y - w = -1 \\ x = 1 \end{cases}$$

$$\rightsquigarrow \begin{cases} x = 1 \\ y = -1 + t \\ z = 3 - t \\ w = t \end{cases}$$

$t = w$ is a "free variable".

This system has ∞'ly many solutions, on for each $t \in \mathbb{R}$.